Digital Communication Through Band-Limited Channels

Channel Characterization, Signal Design, and the Nyquist Criterion

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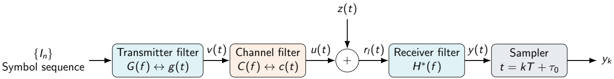
Why band limitation matters

Digital symbols are conveyed by shifted pulses. A physical channel passes only a finite frequency band and may also distort amplitude and phase.

Central question

How fast can symbols be sent through bandwidth W while ensuring that neighboring pulses do not corrupt the decision sample?



$$v(t) = \sum_{n=0}^{\infty} I_n g(t - nT),$$

Equivalent low-pass channel model

For a bandpass signal

$$s(t) = \Re\{v(t)e^{j2\pi f_c t}\},$$

the equivalent low-pass received signal is

$$r(t) = \int_{-\infty}^{\infty} v(\tau)c(t-\tau)\,d\tau + z(t) = v(t) * c(t) + z(t).$$

In frequency domain, the useful signal becomes $V(f)C(f)$. For a channel of bandwidth W,

$$C(f) = 0, \qquad |f| > W.$$

Thus components of $V(f)$ outside $[-W, W]$ cannot reach the receiver.

Amplitude and delay distortion

Write the channel response as

$$C(f) = |C(f)|e^{j\theta(f)}, \qquad \tau(f) = -\frac{1}{2\pi}\frac{d\theta(f)}{df}.$$

Nondistorting channel

$|C(f)|$ is constant and $\theta(f)$ is linear over $|f| \leq W$; equivalently, envelope delay $\tau(f)$ is constant.

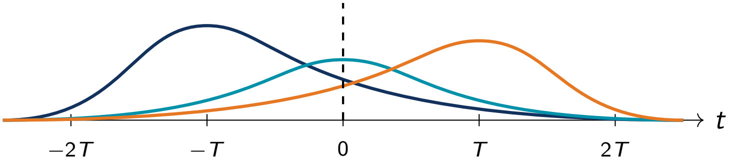
Nonideal channel

Nonflat magnitude causes amplitude distortion; nonlinear phase causes delay distortion. Pulse spreading produces intersymbol interference (ISI).

Other practical impairments include Gaussian/impulse noise, frequency offset, phase jitter, and nonlinear distortion.

How pulse spreading creates ISI

sample for central symbol



tails from adjacent pulses add at the decision instant

Decision sample

The receiver sees desired symbol + neighboring-symbol contributions + noise.

Signal design model: transmitter and channel

The equivalent low-pass signal for many digital modulation methods has the form

$$v(t) = \sum_{n=0}^{\infty} I_n g(t - nT), \tag{9.2-1}$$

where $\{I_n\}$ is the information-bearing symbol sequence and $g(t)$ is a band-limited pulse:

$$G(f) = 0, \qquad |f| > W.$$

If the channel response is $C(f)$, also limited to $|f| \leq W$, the received equivalent low-pass signal is

$$r_l(t) = \sum_{n=0}^{\infty} I_n h(t - nT) + z(t), \tag{9.2-2}$$

where $z(t)$ is additive white Gaussian noise and

$$h(t) = \int_{-\infty}^{\infty} g(\tau) c(t - \tau)\, d\tau = g(t) * c(t). \tag{9.2-3}$$

Signal design model: receiver-filter output

The received signal is passed through a filter and then sampled at $1/T$ samples/s. For optimum signal detection, the receiving filter is matched to the received pulse and has response $H^*(f)$.

Denote its output by

$$y(t) = \sum_{n=0}^{\infty} I_n x(t - nT) + \nu(t), \qquad (9.2\text{-}4)$$

where $x(t)$ is the response of the receiving filter to $h(t)$ and $\nu(t)$ is the filtered version of the channel noise $z(t)$.

Sampling at $t = kT + \tau_0$ gives

$$y(kT + \tau_0) \equiv y_k = \sum_{n=0}^{\infty} I_n x(kT - nT + \tau_0) + \nu(kT + \tau_0). \qquad (9.2\text{-}5)$$

Here τ_0 is the transmission delay through the channel.

Sampled model: desired symbol, ISI, and noise

Define the sampled pulse and sampled noise as

$$x_{k-n} \equiv x\big((k-n)T + \tau_0\big), \qquad \nu_k \equiv \nu(kT + \tau_0).$$

Equation (9.2-5) becomes

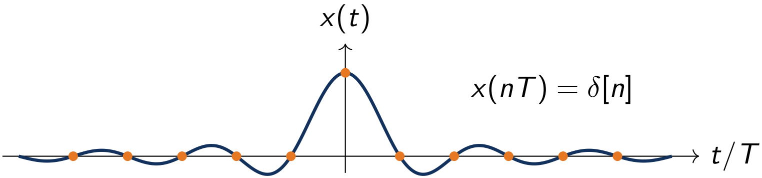
$$y_k = \sum_{n=0}^{\infty} I_n x_{k-n} + \nu_k, \qquad k = 0, 1, \ldots \tag{9.2-6}$$

Separating the desired term ($n = k$) from all other symbols,

$$y_k = x_0 \left(I_k + \frac{1}{x_0} \sum_{\substack{n=0 \\ n \neq k}}^{\infty} I_n x_{k-n} \right) + \nu_k, \qquad k = 0, 1, \ldots \tag{9.2-7}$$

Thus $x_0 I_k$ is the desired contribution, the summation is ISI, and ν_k is additive noise. With the convenient normalization $x_0 = 1$, zero ISI requires $x_m = 0$ for every nonzero integer m.

Physical meaning of the zero-ISI condition



- Pulses may overlap between samples; they need not be time-limited.
- At the sampling grid, every unwanted shifted pulse is exactly at a zero.
- Zero ISI does not mean zero noise or immunity to sampling-time error.

Nyquist zero-ISI theorem

Theorem (Nyquist)

Let $x(t) \leftrightarrow X(f)$ *and let the sampling interval be* T. *Then*

$$x(nT) = \delta[n] \quad \textit{for all } n \in \mathbb{Z}$$

if and only if

$$\boxed{\sum_{m=-\infty}^{\infty} X\left(f + \frac{m}{T}\right) = T}$$

for (almost) every f *in any interval of length* $1/T$.

Intuition

Shift the spectrum by every integer multiple of $1/T$. Zero ISI occurs exactly when these replicas add to a flat constant. Spectral gaps or ripple translate into nonzero samples at nT.

Proof (1/4): partition the Fourier integral

Start with the inverse Fourier transform at $t = nT$:

$$x(nT) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f nT}\, df.$$

Partition the frequency axis into adjacent cells of width $1/T$:

$$x(nT) = \sum_{m\in\mathbb{Z}} \int_{(2m-1)/(2T)}^{(2m+1)/(2T)} X(f) e^{j2\pi f nT}\, df.$$

In the mth cell substitute $f = u + m/T$. Since $n, m \in \mathbb{Z}$,

$$e^{j2\pi(u+m/T)nT} = e^{j2\pi unT} e^{j2\pi mn} = e^{j2\pi unT}.$$

This integer-phase cancellation is the key step.

Proof (2/4): form the periodic alias sum

After the substitution, every integral has the same limits:

$$\begin{aligned} x(nT) &= \int_{-1/(2T)}^{1/(2T)} \left[\sum_{m \in \mathbb{Z}} X\Big(u + \frac{m}{T}\Big) \right] e^{j2\pi unT}\, du \\ &= \int_{-1/(2T)}^{1/(2T)} B(u) e^{j2\pi unT}\, du, \end{aligned}$$

where

$$B(f) = \sum_{m \in \mathbb{Z}} X\Big(f + \frac{m}{T}\Big).$$

$B(f)$ is periodic with period $1/T$: shifting f by $1/T$ only reindexes the sum.

Proof (3/4): connect samples to Fourier coefficients

Expand the $1/T$-periodic function $B(f)$ in a Fourier series:

$$B(f) = \sum_{k\in\mathbb{Z}} b_k e^{j2\pi kfT},$$

with coefficients

$$b_k = T\int_{-1/(2T)}^{1/(2T)} B(f)e^{-j2\pi kfT}\,df.$$

Comparing with the expression for $x(nT)$ gives

$$\boxed{b_k = T\,x(-kT)}.$$

Therefore the complete set of time samples $\{x(nT)\}$ is, up to scale and reversal, exactly the set of Fourier-series coefficients of the alias sum $B(f)$.

Proof (4/4): necessity and sufficiency

Necessity. If $x(nT) = \delta[n]$, then

$$b_k = T\delta[k].$$

Substitution into the Fourier series gives

$$B(f) = T.$$

Sufficiency. Conversely, if $B(f) = T$, its Fourier coefficients are

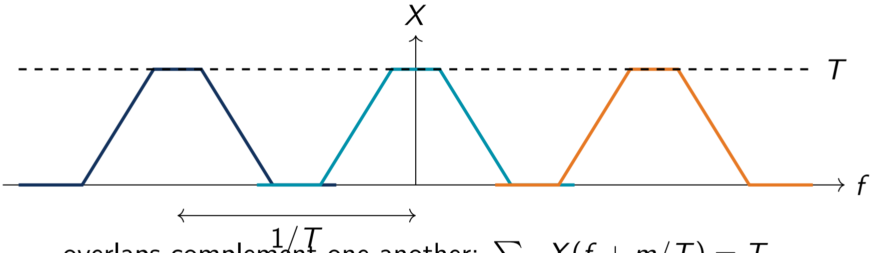
$$b_0 = T, \qquad b_k = 0 \ (k \neq 0).$$

Since $b_k = Tx(-kT)$, it follows that

$$x(nT) = \delta[n].$$

Both directions are proved; hence the condition is necessary and sufficient. $\square$

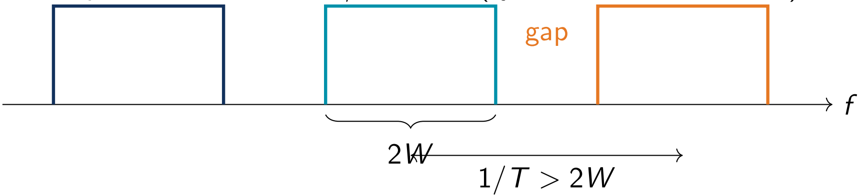
Visual intuition: replicas must tile to a constant



overlaps complement one another: $\sum_m X(f + m/T) = T$

Case 1: $T < \frac{1}{2W}$ – impossible

Equivalent condition: $1/T > 2W$ (symbol rate exceeds $2W$).

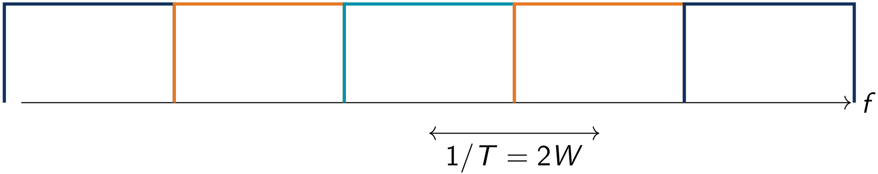


Reason

The replicas do not overlap, leaving spectral gaps in $B(f)$. A constant $B(f) = T$ is impossible; no band-limited zero-ISI pulse exists.

Case 2: $T = \frac{1}{2W}$ – Nyquist limit

Equivalent condition: $1/T = 2W$.



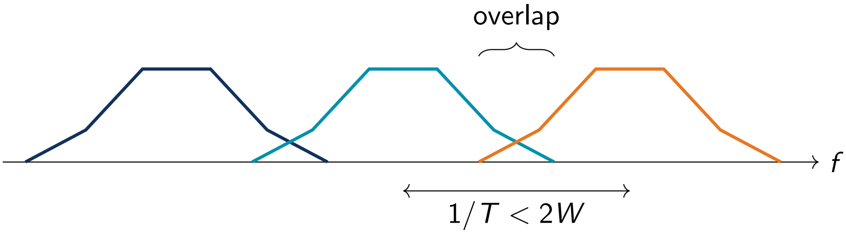
Only the brick-wall spectrum tiles the axis:

$$X(f) = \begin{cases} T, & |f| < W, \\ 0, & \text{otherwise}, \end{cases} \qquad x(t) = \frac{\sin(\pi t/T)}{\pi t/T} = \mathrm{sinc}(t/T).$$

The ideal sinc is noncausal and its $1/t$ tails make it sensitive to timing errors and truncation.

Case 3: $T > \frac{1}{2W}$ – many solutions

Equivalent condition: $1/T < 2W$ (rate below $2W$).



Design freedom

Overlapping transition bands can be shaped so neighboring replicas add to T. Smooth roll-off gives faster time-domain decay and better robustness.

The three cases at a glance

Relation	Symbol rate $R_s = 1/T$	Zero-ISI pulse?	Interpretation
$T < 1/(2W)$	$R_s > 2W$	No	Replica gaps prevent constant alias sum
$T = 1/(2W)$	$R_s = 2W$	Unique ideal choice	Rectangular $X(f)$, sinc $x(t)$
$T > 1/(2W)$	$R_s < 2W$	Many choices	Replica overlap enables smooth roll-off

Nyquist rate for real baseband bandwidth W

The maximum zero-ISI symbol rate is $2W$ symbols/s. For M-ary signaling, the corresponding ideal bit rate is $R_b = 2W \log_2 M$ bit/s.

Raised-cosine spectrum

For $0 \leq \beta \leq 1$,

$$X_{RC}(f) = \begin{cases} T, & 0 \leq |f| \leq \dfrac{1-\beta}{2T}, \\ \dfrac{T}{2}\left[1 + \cos\left(\dfrac{\pi T}{\beta}\left(|f| - \dfrac{1-\beta}{2T}\right)\right)\right], & \dfrac{1-\beta}{2T} \leq |f| \leq \dfrac{1+\beta}{2T}, \\ 0, & |f| > \dfrac{1+\beta}{2T}. \end{cases}$$

Bandwidth and symbol rate satisfy

$$W = \frac{1+\beta}{2T}, \qquad R_s = \frac{1}{T} = \frac{2W}{1+\beta}.$$

β is the roll-off factor: more excess bandwidth buys a smoother spectrum and a better-behaved pulse.

Raised-cosine pulse and practical meaning

$$x_{RC}(t) = \operatorname{sinc}(t/T)\,\frac{\cos(\pi\beta t/T)}{1 - 4\beta^2 t^2/T^2}.$$

At apparent singularities $t = \pm T/(2\beta)$ the limiting value is finite.

$\beta = 0$

Minimum bandwidth, sinc pulse, tails decay as $1/t$.

$\beta > 0$

Extra bandwidth; tails decay as $1/t^3$, improving truncation and timing-error behavior.

In an ideal channel, split $X_{RC}(f)$ between matched transmit and receive filters: each uses a root-raised-cosine magnitude so their product is raised cosine.

Numerical example 1: classify the three regimes

Let $W = 3\,\text{kHz}$. The limiting interval is

$$T_{\min} = \frac{1}{2W} = \frac{1}{6000} = 166.7\,\mu\text{s}.$$

1. $T = 125\,\mu$s: $R_s = 8$ ksym/s > 6 ksym/s. Impossible to achieve zero ISI with bandwidth 3 kHz.
2. $T = 166.7\,\mu$s: $R_s = 6$ ksym/s. Nyquist-limit case; ideal spectrum is rectangular and pulse is sinc.
3. $T = 250\,\mu$s: $R_s = 4$ ksym/s. Many zero-ISI spectra are possible; a raised cosine can be used.

Key takeaways

1. Band-limited channels smear pulses; magnitude and delay distortion create ISI.
2. Zero ISI requires $x(nT) = \delta[n]$: overlap is allowed, but unwanted pulses must cross zero at each decision time.
3. The Nyquist theorem is equivalent in frequency domain to

$$\sum_m X(f + m/T) = T.$$

4. Zero-ISI transmission is impossible for $T < 1/(2W)$, unique and sinc-shaped at equality, and admits many practical pulse shapes for $T > 1/(2W)$.
5. Raised-cosine shaping trades excess bandwidth for smoother filtering and improved time-domain decay.

Reference and scope

J. G. Proakis and M. Salehi, *Digital Communications*, Chapter 9, Sections 9.1, 9.2, and 9.2-1.

Questions?